technische universität
dortmund

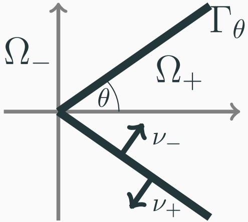
Spectral properties of Schrödinger operators with δ'-interactions via Robin Laplacians

Marco Vogel

Joint work with Konstantin Pankrashkin

Problem setting

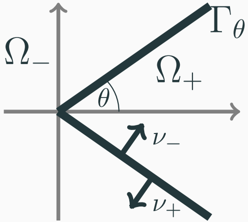
Consider a star graph $\Gamma_\theta \subset \mathbb{R}^2$ with two branches



Problem setting

Consider a star graph $\Gamma_\theta \subset \mathbb{R}^2$ with two branches and the quadratic form

$$h_\theta[u] = \int_{\mathbb{R}^2 \setminus \Gamma_\theta} |\nabla u|^2 \mathrm{d}x - \int_{\Gamma_\theta} |u_+ - u_-|^2 \mathrm{d}\sigma, \quad D(h_\theta) = H^1(\mathbb{R}^2 \setminus \Gamma_\theta).$$

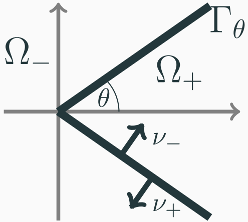


Problem setting

Consider a star graph $\Gamma_\theta \subset \mathbb{R}^2$ with two branches and the quadratic form

$$h_\theta[u] = \int_{\mathbb{R}^2 \setminus \Gamma_\theta} |\nabla u|^2 \mathrm{d}x - \int_{\Gamma_\theta} |u_+ - u_-|^2 \mathrm{d}\sigma, \quad D(h_\theta) = H^1(\mathbb{R}^2 \setminus \Gamma_\theta).$$

The self-adjoint operator H_θ associated with h_θ is called a Schrödinger operator with a δ'-interaction supported on Γ_θ.



Problem setting

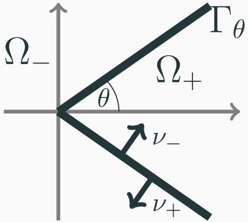
Consider a star graph $\Gamma_\theta \subset \mathbb{R}^2$ with two branches and the quadratic form

$$h_\theta[u] = \int_{\mathbb{R}^2 \setminus \Gamma_\theta} |\nabla u|^2 \mathrm{d}x - \int_{\Gamma_\theta} |u_+ - u_-|^2 \mathrm{d}\sigma, \quad D(h_\theta) = H^1(\mathbb{R}^2 \setminus \Gamma_\theta).$$

The self-adjoint operator H_θ associated with h_θ is called a Schrödinger operator with a δ'-interaction supported on Γ_θ.

Informally, this operator can be viewed as

$$\begin{aligned} H_\theta u &= -\Delta u \\ \partial_{\nu_+} u_+ &= -\partial_{\nu_-} u_- \qquad \text{on } \Gamma_\theta \\ \partial_{\nu_+} u_+ &= u_+ - u_- \qquad \text{on } \Gamma_\theta \end{aligned}$$



where $\partial_{\nu_\pm}$ is the outward normal derivative.

Tools

- Variational eigenvalues and Min-Max principle:

$$\Lambda_j(H_\theta) := \inf_{\substack{V \subset H^1(\mathbb{R}^2 \setminus \Gamma_\theta) \\ \dim V = j}} \sup_{\substack{u \in V \\ u \neq 0}} \frac{h_\theta[u]}{\|u\|^2_{L^2(\mathbb{R}^2 \setminus \Gamma_\theta)}}$$

Tools

- Variational eigenvalues and Min-Max principle:

$$\Lambda_j(H_\theta) := \inf_{\substack{V \subset H^1(\mathbb{R}^2 \setminus \Gamma_\theta) \\ \dim V = j}} \sup_{\substack{u \in V \\ u \neq 0}} \frac{h_\theta[u]}{\|u\|^2_{L^2(\mathbb{R}^2 \setminus \Gamma_\theta)}}$$

- Comparison with other Operators:

	Robin	Schrödinger with δ-interaction	Schrödinger with δ'-interaction
Quadr. Form	$r_{\theta,\alpha}[u] = \int_{\Omega_+} \lvert\nabla u\rvert^2 \, \mathrm{d}x - \alpha \int_{\Gamma_\theta} \lvert u\rvert^2 \, \mathrm{d}\sigma$	$d_{\theta,\alpha}[u] = \int_{\mathbb{R}^2} \lvert\nabla u\rvert^2 \, \mathrm{d}x - \alpha \int_{\Gamma_\theta} \lvert u\rvert^2 \mathrm{d}\sigma$	$h_{\theta,\alpha}[u] = \int_{\mathbb{R}^2 \setminus \Gamma_\theta} \lvert\nabla u\rvert^2 \, \mathrm{d}x - \alpha \int_{\Gamma_\theta} \lvert u_+ - u_-\rvert^2 \, \mathrm{d}\sigma$

Tools

- Variational eigenvalues and Min-Max principle:

$$\Lambda_j(H_\theta) := \inf_{\substack{V \subset H^1(\mathbb{R}^2 \setminus \Gamma_\theta) \\ \dim V = j}} \sup_{\substack{u \in V \\ u \neq 0}} \frac{h_\theta[u]}{\|u\|^2_{L^2(\mathbb{R}^2 \setminus \Gamma_\theta)}}$$

- Comparison with other Operators:

	Robin	Schrödinger with δ-interaction	Schrödinger with δ'-interaction
Quadr. Form	$r_{\theta,\alpha}[u] = \int_{\Omega_+} \|\nabla u\|^2 \, \mathrm{d}x - \alpha \int_{\Gamma_\theta} \|u\|^2 \, \mathrm{d}\sigma$	$d_{\theta,\alpha}[u] = \int_{\mathbb{R}^2} \|\nabla u\|^2 \, \mathrm{d}x - \alpha \int_{\Gamma_\theta} \|u\|^2 \mathrm{d}\sigma$	$h_{\theta,\alpha}[u] = \int_{\mathbb{R}^2 \setminus \Gamma_\theta} \|\nabla u\|^2 \, \mathrm{d}x - \alpha \int_{\Gamma_\theta} \|u_+ - u_-\|^2 \, \mathrm{d}\sigma$
Form domain	$D(r_{\theta,\alpha}) = H^1(\Omega_+)$	$D(d_{\theta,\alpha}) = H^1(\mathbb{R}^2)$	$D(h_{\theta,\alpha}) = H^1(\mathbb{R}^2 \setminus \Gamma_\theta)$

Tools

- Variational eigenvalues and Min-Max principle:

$$\Lambda_j(H_\theta) := \inf_{\substack{V \subset H^1(\mathbb{R}^2 \setminus \Gamma_\theta) \\ \dim V = j}} \sup_{\substack{u \in V \\ u \neq 0}} \frac{h_\theta[u]}{\|u\|^2_{L^2(\mathbb{R}^2 \setminus \Gamma_\theta)}}$$

- Comparison with other Operators:

	Robin	Schrödinger with δ-interaction	Schrödinger with δ'-interaction
Quadr. Form	$r_{\theta,\alpha}[u] = \int_{\Omega_+} \|\nabla u\|^2 \, \mathrm{d}x - \alpha \int_{\Gamma_\theta} \|u\|^2 \, \mathrm{d}\sigma$	$d_{\theta,\alpha}[u] = \int_{\mathbb{R}^2} \|\nabla u\|^2 \, \mathrm{d}x - \alpha \int_{\Gamma_\theta} \|u\|^2 \mathrm{d}\sigma$	$h_{\theta,\alpha}[u] = \int_{\mathbb{R}^2 \setminus \Gamma_\theta} \|\nabla u\|^2 \, \mathrm{d}x - \alpha \int_{\Gamma_\theta} \|u_+ - u_-\|^2 \, \mathrm{d}\sigma$
Form domain	$D(r_{\theta,\alpha}) = H^1(\Omega_+)$	$D(d_{\theta,\alpha}) = H^1(\mathbb{R}^2)$	$D(h_{\theta,\alpha}) = H^1(\mathbb{R}^2 \setminus \Gamma_\theta)$
Operator	$R_{\theta,\alpha} u = -\Delta u$	$D_{\theta,\alpha} u = -\Delta u$	$H_{\theta,\alpha} u = -\Delta u$

Tools

- Variational eigenvalues and Min-Max principle:

$$\Lambda_j(H_\theta) := \inf_{\substack{V \subset H^1(\mathbb{R}^2 \setminus \Gamma_\theta) \\ \dim V = j}} \sup_{\substack{u \in V \\ u \neq 0}} \frac{h_\theta[u]}{\|u\|^2_{L^2(\mathbb{R}^2 \setminus \Gamma_\theta)}}$$

- Comparison with other Operators:

	Robin	Schrödinger with δ-interaction	Schrödinger with δ'-interaction
Quadr. Form	$r_{\theta,\alpha}[u] = \int_{\Omega_+} \|\nabla u\|^2 \, \mathrm{d}x - \alpha \int_{\Gamma_\theta} \|u\|^2 \, \mathrm{d}\sigma$	$d_{\theta,\alpha}[u] = \int_{\mathbb{R}^2} \|\nabla u\|^2 \, \mathrm{d}x - \alpha \int_{\Gamma_\theta} \|u\|^2 \mathrm{d}\sigma$	$h_{\theta,\alpha}[u] = \int_{\mathbb{R}^2 \setminus \Gamma_\theta} \|\nabla u\|^2 \, \mathrm{d}x - \alpha \int_{\Gamma_\theta} \|u_+ - u_-\|^2 \, \mathrm{d}\sigma$
Form domain	$D(r_{\theta,\alpha}) = H^1(\Omega_+)$	$D(d_{\theta,\alpha}) = H^1(\mathbb{R}^2)$	$D(h_{\theta,\alpha}) = H^1(\mathbb{R}^2 \setminus \Gamma_\theta)$
Operator	$R_{\theta,\alpha} u = -\Delta u$	$D_{\theta,\alpha} u = -\Delta u$	$H_{\theta,\alpha} u = -\Delta u$
Bdry. conditions	$\alpha u = \partial_{\nu_+} u$	$\alpha u_+ = \partial_{\nu_-} u_- + \partial_{\nu_+} u_+$	$\alpha \partial_{\nu_+} u_+ = u_+ - u_-, \quad \partial_{\nu_+} u_+ = -\partial_{\nu_-} u_-$

Tools

- Variational eigenvalues and Min-Max principle:

$$\Lambda_j(H_\theta) := \inf_{\substack{V \subset H^1(\mathbb{R}^2 \setminus \Gamma_\theta) \\ \dim V = j}} \sup_{\substack{u \in V \\ u \neq 0}} \frac{h_\theta[u]}{\|u\|^2_{L^2(\mathbb{R}^2 \setminus \Gamma_\theta)}}$$

- Comparison with other Operators:

	Robin	Schrödinger with δ-interaction	Schrödinger with δ'-interaction
Quadr. Form	$r_{\theta,\alpha}[u] = \int_{\Omega_+} \lvert\nabla u\rvert^2 \, \mathrm{d}x - \alpha \int_{\Gamma_\theta} \lvert u\rvert^2 \, \mathrm{d}\sigma$	$d_{\theta,\alpha}[u] = \int_{\mathbb{R}^2} \lvert\nabla u\rvert^2 \, \mathrm{d}x - \alpha \int_{\Gamma_\theta} \lvert u\rvert^2 \mathrm{d}\sigma$	$h_{\theta,\alpha}[u] = \int_{\mathbb{R}^2 \setminus \Gamma_\theta} \lvert\nabla u\rvert^2 \, \mathrm{d}x - \alpha \int_{\Gamma_\theta} \lvert u_+ - u_-\rvert^2 \, \mathrm{d}\sigma$
Form domain	$D(r_{\theta,\alpha}) = H^1(\Omega_+)$	$D(d_{\theta,\alpha}) = H^1(\mathbb{R}^2)$	$D(h_{\theta,\alpha}) = H^1(\mathbb{R}^2 \setminus \Gamma_\theta)$
Operator	$R_{\theta,\alpha} u = -\Delta u$	$D_{\theta,\alpha} u = -\Delta u$	$H_{\theta,\alpha} u = -\Delta u$
Bdry. conditions	$\alpha u = \partial_{\nu_+} u$	$\alpha u_+ = \partial_{\nu_-} u_- + \partial_{\nu_+} u_+$	$\alpha \partial_{\nu_+} u_+ = u_+ - u_-, \quad \partial_{\nu_+} u_+ = -\partial_{\nu_-} u_-$
Essential spectrum	$[-\alpha^2, \infty)$	$[-\frac{1}{4}\alpha^2, \infty)$	$[-4\alpha^2, \infty)$

Essential spectrum + Sketch of proof

$$\operatorname{spec}_{\mathrm{ess}} H_\theta = [-4, \infty)$$

Essential spectrum + Sketch of proof

$\mathrm{spec}_{\mathrm{ess}}\, H_\theta = [-4, \infty)$

"$\subset$": Use $|u_+ - u_-|^2 \leq 2(|u_+|^2 + |u_-|^2)$ to estimate

Essential spectrum + Sketch of proof

$\mathrm{spec}_{\mathrm{ess}} H_\theta = [-4, \infty)$

"$\subset$": Use $|u_+ - u_-|^2 \le 2(|u_+|^2 + |u_-|^2)$ to estimate

$$h_\theta[u] = \int_{\mathbb{R}^2 \setminus \Gamma_\theta} |\nabla u|^2 \mathrm{d}x - \int_{\Gamma_\theta} |u_+ - u_-|^2 \mathrm{d}\sigma$$

Essential spectrum + Sketch of proof

$\mathrm{spec}_{\mathrm{ess}}\, H_\theta = [-4, \infty)$

"$\subset$": Use $|u_+ - u_-|^2 \leq 2(|u_+|^2 + |u_-|^2)$ to estimate

$$\begin{aligned} h_\theta[u] &= \int_{\mathbb{R}^2 \setminus \Gamma_\theta} |\nabla u|^2 \mathrm{d}x - \int_{\Gamma_\theta} |u_+ - u_-|^2 \mathrm{d}\sigma \\ &\geq \int_{\Omega_+} |\nabla u_+|^2 \mathrm{d}x - 2\int_{\Gamma_\theta} |u_+|^2 \mathrm{d}\sigma + \int_{\Omega_-} |\nabla u_-|^2 \mathrm{d}x - 2\int_{\Gamma_\theta} |u_-|^2 \mathrm{d}\sigma = r_{\theta,2}[u_+] + r_{\pi-\theta,2}[u_-] \end{aligned}$$

Essential spectrum + Sketch of proof

$\mathrm{spec}_{\mathrm{ess}}\, H_\theta = [-4, \infty)$

"$\subset$": Use $|u_+ - u_-|^2 \leq 2(|u_+|^2 + |u_-|^2)$ to estimate

$$
\begin{aligned}
h_\theta[u] &= \int_{\mathbb{R}^2 \setminus \Gamma_\theta} |\nabla u|^2 \mathrm{d}x - \int_{\Gamma_\theta} |u_+ - u_-|^2 \mathrm{d}\sigma \\
&\geq \int_{\Omega_+} |\nabla u_+|^2 \mathrm{d}x - 2\int_{\Gamma_\theta} |u_+|^2 \mathrm{d}\sigma + \int_{\Omega_-} |\nabla u_-|^2 \mathrm{d}x - 2\int_{\Gamma_\theta} |u_-|^2 \mathrm{d}\sigma = r_{\theta,2}[u_+] + r_{\pi-\theta,2}[u_-]
\end{aligned}
$$

The RHS is the sum of two Robin-Laplacians $R_{\theta,2}$ and $R_{\pi-\theta,2}$.

Essential spectrum + Sketch of proof

$\operatorname{spec}_{\mathrm{ess}} H_\theta = [-4, \infty)$

"$\subset$": Use $|u_+ - u_-|^2 \leq 2(|u_+|^2 + |u_-|^2)$ to estimate

$$h_\theta[u] = \int_{\mathbb{R}^2\setminus\Gamma_\theta} |\nabla u|^2 \mathrm{d}x - \int_{\Gamma_\theta} |u_+ - u_-|^2 \mathrm{d}\sigma$$
$$\geq \int_{\Omega_+} |\nabla u_+|^2 \mathrm{d}x - 2\int_{\Gamma_\theta} |u_+|^2 \mathrm{d}\sigma + \int_{\Omega_-} |\nabla u_-|^2 \mathrm{d}x - 2\int_{\Gamma_\theta} |u_-|^2 \mathrm{d}\sigma = r_{\theta,2}[u_+] + r_{\pi-\theta,2}[u_-]$$

The RHS is the sum of two Robin-Laplacians $R_{\theta,2}$ and $R_{\pi-\theta,2}$.

$\operatorname{spec}_{\mathrm{ess}} R_{\varphi,\alpha} = [-\alpha^2, \infty)$ (Khalile & Pankrashkin '2018)

Essential spectrum + Sketch of proof

$\mathrm{spec_{ess}}\, H_\theta = [-4, \infty)$

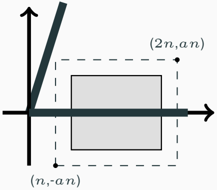
"$\subset$": Use $|u_+ - u_-|^2 \leq 2(|u_+|^2 + |u_-|^2)$ to estimate

$$h_\theta[u] = \int_{\mathbb{R}^2\setminus\Gamma_\theta} |\nabla u|^2 \mathrm{d}x - \int_{\Gamma_\theta} |u_+ - u_-|^2 \mathrm{d}\sigma$$
$$\geq \int_{\Omega_+} |\nabla u_+|^2 \mathrm{d}x - 2\int_{\Gamma_\theta} |u_+|^2 \mathrm{d}\sigma + \int_{\Omega_-} |\nabla u_-|^2 \mathrm{d}x - 2\int_{\Gamma_\theta} |u_-|^2 \mathrm{d}\sigma = r_{\theta,2}[u_+] + r_{\pi-\theta,2}[u_-]$$

The RHS is the sum of two Robin-Laplacians $R_{\theta,2}$ and $R_{\pi-\theta,2}$.

$$\mathrm{spec_{ess}}\, R_{\varphi,\alpha} = [-\alpha^2, \infty) \quad \text{(Khalile \& Pankrashkin '2018)}$$

"$\supset$": $f_n(x,y) := \psi_1(y) e^{ikx} \chi_n(x) \widetilde{\chi}_n(y)$ is a Weyl sequence for $k^2 - 4$.



Cardinality of the discrete spectrum + Sketch of proof

$$|\operatorname{spec}_{\mathrm{disc}} H_\theta| < \infty$$

Proofsketch:

- $\Lambda_j(H_\theta) \geq \Lambda_j(R_{\theta,2} \oplus R_{\pi-\theta,2})$

Cardinality of the discrete spectrum + Sketch of proof

$$|\operatorname{spec}_{\text{disc}} H_\theta| < \infty$$

Proofsketch:

- $\Lambda_j(H_\theta) \geq \Lambda_j(R_{\theta,2} \oplus R_{\pi-\theta,2})$
- $\operatorname{spec}_{\text{ess}} H_\theta = \operatorname{spec}_{\text{ess}} R_{\theta,2} \oplus R_{\pi-\theta,2} = [-4, \infty)$

Cardinality of the discrete spectrum + Sketch of proof

$$|\operatorname{spec}_{\mathrm{disc}} H_\theta| < \infty$$

Proofsketch:

- $\Lambda_j(H_\theta) \geq \Lambda_j(R_{\theta,2} \oplus R_{\pi-\theta,2})$
- $\operatorname{spec}_{\mathrm{ess}} H_\theta = \operatorname{spec}_{\mathrm{ess}} R_{\theta,2} \oplus R_{\pi-\theta,2} = [-4, \infty)$
- $|\operatorname{spec}_{\mathrm{disc}} R_{\theta,2}| < \infty$ and $\operatorname{spec}_{\mathrm{disc}} R_{\pi-\theta,2} = \emptyset$ (Khalile & Pankrashkin '2018)

Cardinality of the discrete spectrum + Sketch of proof

$$|\operatorname{spec}_{\mathrm{disc}} H_\theta| < \infty$$

Proofsketch:

- $\Lambda_j(H_\theta) \geq \Lambda_j(R_{\theta,2} \oplus R_{\pi-\theta,2})$
- $\operatorname{spec}_{\mathrm{ess}} H_\theta = \operatorname{spec}_{\mathrm{ess}} R_{\theta,2} \oplus R_{\pi-\theta,2} = [-4, \infty)$
- $|\operatorname{spec}_{\mathrm{disc}} R_{\theta,2}| < \infty$ and $\operatorname{spec}_{\mathrm{disc}} R_{\pi-\theta,2} = \emptyset$ (Khalile & Pankrashkin '2018)
- $|\operatorname{spec}_{\mathrm{disc}} H_\theta| < \infty$

Cardinality of the discrete spectrum II + Sketch of proof

$$\operatorname{spec}_{\text{disc}} H_\theta \neq \emptyset \iff \theta \neq \frac{\pi}{2}$$

Proofsketch: "$\Leftarrow$"

- $D_{\theta,\alpha}$ has a discrete eigenvalue with eigenfunction $\widetilde{u} \in H^1(\mathbb{R}^2)$. (Exner & Ichinose '2001)

Cardinality of the discrete spectrum II + Sketch of proof

$$\operatorname{spec}_{\mathrm{disc}} H_\theta \neq \emptyset \iff \theta \neq \frac{\pi}{2}$$

Proofsketch: "$\Leftarrow$"

- $D_{\theta,\alpha}$ has a discrete eigenvalue with eigenfunction $\widetilde{u} \in H^1(\mathbb{R}^2)$. (Exner & Ichinose '2001)
- $\operatorname{spec}_{\mathrm{ess}} H_\theta = [-4, \infty) = \operatorname{spec}_{\mathrm{ess}} D_{\theta,4}$ (Exner & Ichinose '2001)

Cardinality of the discrete spectrum II + Sketch of proof

$$\operatorname{spec}_{\text{disc}} H_\theta \neq \emptyset \iff \theta \neq \frac{\pi}{2}$$

Proofsketch: "$\Leftarrow$"

- $D_{\theta,\alpha}$ has a discrete eigenvalue with eigenfunction $\widetilde{u} \in H^1(\mathbb{R}^2)$. (Exner & Ichinose '2001)
- $\operatorname{spec}_{\text{ess}} H_\theta = [-4, \infty) = \operatorname{spec}_{\text{ess}} D_{\theta,4}$ (Exner & Ichinose '2001)
- For $u := (1_{\Omega_+} - 1_{\Omega_-})\widetilde{u} \in H^1(\mathbb{R}^2 \setminus \Gamma_\theta)$

Cardinality of the discrete spectrum II + Sketch of proof

$$\operatorname{spec}_{\mathrm{disc}} H_\theta \neq \emptyset \iff \theta \neq \frac{\pi}{2}$$

Proofsketch: "$\Leftarrow$"

- $D_{\theta,\alpha}$ has a discrete eigenvalue with eigenfunction $\widetilde{u} \in H^1(\mathbb{R}^2)$. (Exner & Ichinose '2001)
- $\operatorname{spec}_{\mathrm{ess}} H_\theta = [-4, \infty) = \operatorname{spec}_{\mathrm{ess}} D_{\theta,4}$ (Exner & Ichinose '2001)
- For $u := (1_{\Omega_+} - 1_{\Omega_-})\widetilde{u} \in H^1(\mathbb{R}^2 \setminus \Gamma_\theta)$ we have

$$h_\theta[u] = \int_{\mathbb{R}^2\setminus\Gamma_\theta} |\nabla u|^2 \mathrm{d}x - \int_{\Gamma_\theta} |u_+ - u_-|^2 \mathrm{d}\sigma = \int_{\mathbb{R}^2} |\nabla \widetilde{u}|^2 \mathrm{d}x - \int_{\Gamma_\theta} |\widetilde{u}_+ + \widetilde{u}_-|^2 \mathrm{d}\sigma$$

Cardinality of the discrete spectrum II + Sketch of proof

$$\operatorname{spec}_{\mathrm{disc}} H_\theta \neq \emptyset \iff \theta \neq \frac{\pi}{2}$$

Proofsketch: "$\Leftarrow$"

- $D_{\theta,\alpha}$ has a discrete eigenvalue with eigenfunction $\widetilde{u} \in H^1(\mathbb{R}^2)$. (Exner & Ichinose '2001)
- $\operatorname{spec}_{\mathrm{ess}} H_\theta = [-4, \infty) = \operatorname{spec}_{\mathrm{ess}} D_{\theta,4}$ (Exner & Ichinose '2001)
- For $u := (1_{\Omega_+} - 1_{\Omega_-})\widetilde{u} \in H^1(\mathbb{R}^2 \setminus \Gamma_\theta)$ we have

$$\begin{aligned} h_\theta[u] &= \int_{\mathbb{R}^2\setminus\Gamma_\theta} |\nabla u|^2 \mathrm{d}x - \int_{\Gamma_\theta} |u_+ - u_-|^2 \mathrm{d}\sigma = \int_{\mathbb{R}^2} |\nabla \widetilde{u}|^2 \mathrm{d}x - \int_{\Gamma_\theta} |\widetilde{u}_+ + \widetilde{u}_-|^2 \mathrm{d}\sigma \\ &= \int_{\mathbb{R}^2} |\nabla \widetilde{u}|^2 \mathrm{d}x - 4\int_{\Gamma_\theta} |\widetilde{u}|^2 \mathrm{d}\sigma = d_{\theta,4}[\widetilde{u}] < -4\|\widetilde{u}\|^2_{L^2(\mathbb{R}^2)} = -4\|u\|^2_{L^2(\mathbb{R}^2\setminus\Gamma_\theta)} \end{aligned}$$

Cardinality of the discrete spectrum II + Sketch of proof

$$\operatorname{spec}_{\mathrm{disc}} H_\theta \neq \emptyset \iff \theta \neq \frac{\pi}{2}$$

Proofsketch: "$\Leftarrow$"

- $D_{\theta,\alpha}$ has a discrete eigenvalue with eigenfunction $\widetilde{u} \in H^1(\mathbb{R}^2)$. (Exner & Ichinose '2001)
- $\operatorname{spec}_{\mathrm{ess}} H_\theta = [-4, \infty) = \operatorname{spec}_{\mathrm{ess}} D_{\theta,4}$ (Exner & Ichinose '2001)
- For $u := (1_{\Omega_+} - 1_{\Omega_-})\widetilde{u} \in H^1(\mathbb{R}^2 \setminus \Gamma_\theta)$ we have

$$\begin{aligned} h_\theta[u] &= \int_{\mathbb{R}^2\setminus\Gamma_\theta} |\nabla u|^2 \mathrm{d}x - \int_{\Gamma_\theta} |u_+ - u_-|^2 \mathrm{d}\sigma = \int_{\mathbb{R}^2} |\nabla \widetilde{u}|^2 \mathrm{d}x - \int_{\Gamma_\theta} |\widetilde{u}_+ + \widetilde{u}_-|^2 \mathrm{d}\sigma \\ &= \int_{\mathbb{R}^2} |\nabla \widetilde{u}|^2 \mathrm{d}x - 4\int_{\Gamma_\theta} |\widetilde{u}|^2 \mathrm{d}\sigma = d_{\theta,4}[\widetilde{u}] < -4\|\widetilde{u}\|^2_{L^2(\mathbb{R}^2)} = -4\|u\|^2_{L^2(\mathbb{R}^2\setminus\Gamma_\theta)} \end{aligned}$$

- $|\operatorname{spec}_{\mathrm{disc}} H_\theta| \geq 1$

"$\Rightarrow$" separation of variables

Small angle asymptotics

Question: How do the eigenvalues of H_θ behave as $\theta \to 0$?

Small angle asymptotics

Question: How do the eigenvalues of H_θ behave as $\theta \to 0$?

$$\Lambda_j(H_\theta) = -\frac{1}{(2j-1)^2\theta^2} + O(\theta^{-1})$$

Small angle asymptotics

Question: How do the eigenvalues of H_θ behave as $\theta \to 0$?

$$\Lambda_j(H_\theta) = -\frac{1}{(2j-1)^2\theta^2} + O(\theta^{-1})$$

Proofsketch:

$$\Lambda_j(H_\theta) = \inf_{\substack{V \subset H^1(\mathbb{R}^2 \setminus \Gamma_\theta) \\ \dim V = j}} \sup_{\substack{u \in V \\ u \neq 0}} \frac{\int_{\mathbb{R}^2 \setminus \Gamma_\theta} |\nabla u|^2 \, \mathrm{d}x - \int_{\Gamma_\theta} |u_+ - u_-|^2 \, \mathrm{d}\sigma}{\|u\|^2_{L^2(\mathbb{R}^2 \setminus \Gamma_\theta)}}$$

Small angle asymptotics

Question: How do the eigenvalues of H_θ behave as $\theta \to 0$?

$$\Lambda_j(H_\theta) = -\frac{1}{(2j-1)^2\theta^2} + O(\theta^{-1})$$

Proofsketch:

$$\begin{aligned}
\Lambda_j(H_\theta) &= \inf_{\substack{V \subset H^1(\mathbb{R}^2 \setminus \Gamma_\theta) \\ \dim V = j}} \sup_{\substack{u \in V \\ u \neq 0}} \frac{\int_{\mathbb{R}^2 \setminus \Gamma_\theta} |\nabla u|^2 \,\mathrm{d}x - \int_{\Gamma_\theta} |u_+ - u_-|^2 \,\mathrm{d}\sigma}{\|u\|^2_{L^2(\mathbb{R}^2 \setminus \Gamma_\theta)}} \\
&\leq \inf_{\substack{V \subset H^1(\Omega_+) \oplus \{0\} \\ \dim V = j}} \sup_{\substack{u_+ \oplus 0 \in V \\ u_+ \neq 0}} \frac{\int_{\Omega_+} |\nabla u_+|^2 \,\mathrm{d}x - \int_{\Gamma_\theta} |u_+|^2 \,\mathrm{d}\sigma}{\|u_+\|^2_{L^2(\Omega_+)}} = \Lambda_j(R_{\theta,1})
\end{aligned}$$

Small angle asymptotics

Question: How do the eigenvalues of H_θ behave as $\theta \to 0$?

$$\Lambda_j(H_\theta) = -\frac{1}{(2j-1)^2\theta^2} + O(\theta^{-1})$$

Proofsketch:

$$\Lambda_j(H_\theta) = \inf_{\substack{V \subset H^1(\mathbb{R}^2\setminus\Gamma_\theta) \\ \dim V = j}} \sup_{\substack{u \in V \\ u \neq 0}} \frac{\int_{\mathbb{R}^2\setminus\Gamma_\theta} |\nabla u|^2 \,\mathrm{d}x - \int_{\Gamma_\theta} |u_+ - u_-|^2 \,\mathrm{d}\sigma}{\|u\|^2_{L^2(\mathbb{R}^2\setminus\Gamma_\theta)}}$$

$$\leq \inf_{\substack{V \subset H^1(\Omega_+) \oplus \{0\} \\ \dim V = j}} \sup_{\substack{u_+ \oplus 0 \in V \\ u_+ \neq 0}} \frac{\int_{\Omega_+} |\nabla u_+|^2 \,\mathrm{d}x - \int_{\Gamma_\theta} |u_+|^2 \,\mathrm{d}\sigma}{\|u_+\|^2_{L^2(\Omega_+)}} = \Lambda_j(R_{\theta,1})$$

$$\Lambda_j(R_{\theta,1}) = -\frac{1}{(2j-1)^2\theta^2} + O(1)$$ (Khalile & Pankrashkin '2018)

Small angle asymptotics II

Use $|u_+ - u_-|^2 \leq (1+\varepsilon)|u_+|^2 + (1+1/\varepsilon)|u_-|^2$ to estimate:

Small angle asymptotics II

Use $|u_+ - u_-|^2 \leq (1+\varepsilon)|u_+|^2 + (1+1/\varepsilon)|u_-|^2$ to estimate:

$$\Lambda_j(H_\theta) \geq \Lambda_j(R_{\theta,1+c_j\theta} \oplus R_{\pi-\theta,1+\frac{1}{c_j\theta}}) = \min\{\Lambda_j(R_{\theta,1+c_j\theta}), \inf \operatorname{spec}_{\mathrm{ess}} R_{\pi-\theta,1+\frac{1}{c_j\theta}}\}$$

Small angle asymptotics II

Use $|u_+ - u_-|^2 \le (1+\varepsilon)|u_+|^2 + (1+1/\varepsilon)|u_-|^2$ to estimate:

$$\Lambda_j(H_\theta) \ge \Lambda_j(R_{\theta,1+c_j\theta} \oplus R_{\pi-\theta,1+\frac{1}{c_j\theta}}) = \min\{\Lambda_j(R_{\theta,1+c_j\theta}), \inf \operatorname{spec}_{\mathrm{ess}} R_{\pi-\theta,1+\frac{1}{c_j\theta}}\}$$

$$= \min\{(1+c_j\theta)^2\Lambda_j(R_{\theta,1}), -(1+\tfrac{1}{c_j\theta})^2\}$$

Small angle asymptotics II

Use $|u_+ - u_-|^2 \le (1+\varepsilon)|u_+|^2 + (1+1/\varepsilon)|u_-|^2$ to estimate:

$$\Lambda_j(H_\theta) \ge \Lambda_j(R_{\theta,1+c_j\theta} \oplus R_{\pi-\theta,1+\frac{1}{c_j\theta}}) = \min\{\Lambda_j(R_{\theta,1+c_j\theta}), \inf \operatorname{spec}_{\mathrm{ess}} R_{\pi-\theta,1+\frac{1}{c_j\theta}}\}$$

$$= \min\{(1+c_j\theta)^2\Lambda_j(R_{\theta,1}), -(1+\tfrac{1}{c_j\theta})^2\}$$

$$= \min\{(1+c_j\theta)^2(-\tfrac{1}{(2j-1)^2\theta^2} + O(1)), -(1+\tfrac{1}{c_j\theta})^2\}$$

Small angle asymptotics II

Use $|u_+ - u_-|^2 \leq (1+\varepsilon)|u_+|^2 + (1+1/\varepsilon)|u_-|^2$ to estimate:

$$\Lambda_j(H_\theta) \geq \Lambda_j(R_{\theta,1+c_j\theta} \oplus R_{\pi-\theta,1+\frac{1}{c_j\theta}}) = \min\{\Lambda_j(R_{\theta,1+c_j\theta}), \inf \operatorname{spec}_{\mathrm{ess}} R_{\pi-\theta,1+\frac{1}{c_j\theta}}\}$$

$$= \min\{(1+c_j\theta)^2\Lambda_j(R_{\theta,1}), -(1+\tfrac{1}{c_j\theta})^2\}$$

$$= \min\{(1+c_j\theta)^2(-\tfrac{1}{(2j-1)^2\theta^2} + O(1)), -(1+\tfrac{1}{c_j\theta})^2\}$$

$$= -\tfrac{1}{(2j-1)^2\theta^2} + O(\theta^{-1})$$

Small angle asymptotics II

Use $|u_+ - u_-|^2 \leq (1+\varepsilon)|u_+|^2 + (1+1/\varepsilon)|u_-|^2$ to estimate:

$$\Lambda_j(H_\theta) \geq \Lambda_j(R_{\theta,1+c_j\theta} \oplus R_{\pi-\theta,1+\frac{1}{c_j\theta}}) = \min\{\Lambda_j(R_{\theta,1+c_j\theta}), \inf \operatorname{spec}_{\mathrm{ess}} R_{\pi-\theta,1+\frac{1}{c_j\theta}}\}$$

$$= \min\{(1+c_j\theta)^2\Lambda_j(R_{\theta,1}), -(1+\tfrac{1}{c_j\theta})^2\}$$

$$= \min\{(1+c_j\theta)^2(-\tfrac{1}{(2j-1)^2\theta^2} + O(1)), -(1+\tfrac{1}{c_j\theta})^2\}$$

$$= -\tfrac{1}{(2j-1)^2\theta^2} + O(\theta^{-1})$$

$$\Rightarrow \Lambda_j(H_\theta) = -\tfrac{1}{(2j-1)^2\theta^2} + O(\theta^{-1})$$

Thank you for your attention!